The local mapping. The Maximum Modulus Theorem.

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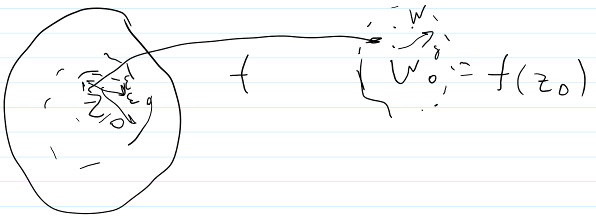
Theorem (local behavior). Assume that $f \in A(\Omega)$, $z_0 \in \Omega$, $f(z_0)=w_0$, and $f(z)-w_0$ has zero of order $n \geq 1$ at z_0 (since $f(z_0)-w_0=0$, $n \geq 1$). Then $\exists \varepsilon_0>0$: $\varepsilon<\varepsilon_0 \Rightarrow \exists \delta>0$: $0<|w-w_0|<\delta \Rightarrow \left(\exists z_1, z_2, \dots, z_n \in B(z,\varepsilon): \forall j: f(z_j)=w \right)$



Proof

$\exists \varepsilon_0>0$: $0<|z-z_0|<\varepsilon_0 \Rightarrow f'(z)\neq 0$. $\forall z: 0<|z-z_0|<\varepsilon_0$: $f(z)\neq w_0$.

Consider $C_\varepsilon = \{|z-z_0|=\varepsilon\}$, $\varepsilon<\varepsilon_0$. $f(z)-w_0\neq 0$ on C_ε.

So $m=\min_{z\in C_\varepsilon} |f(z)-w_0|>0$. positively oriented. Take any w with $|w-w_0|<m$.

$f(z)-w_0$ has n zeroes inside C_ε, counting multiplicity.

$|(f(z)-w)-(f(z)-w_0)| = |w-w_0|<m \leq |f(z)-w_0|$

So, by Rouche, $f(z)-w$ has n zeroes inside $\{|z-z_0|<\varepsilon\}$

All of them are simple ($f'(z)\neq 0$, $z\neq z_0$ in $|z-z_0|<\varepsilon$).

Theorem (analytic maps are open).

Let $\boxed{f\in A(\Omega),\ f\not\equiv \text{const}}$ for some region Ω, $U\subset \Omega$ - open $\Rightarrow f(U)$ - open

Restatement: $\forall z_0\in\Omega$, $\forall\ 0<\varepsilon<\operatorname{dist}(z_0,\partial\Omega)$

$\exists \delta>0$: $\left(|w-f(z_0)|<\delta \Rightarrow \exists z\in B(z_0,\varepsilon): f(z)=w\right)$

$\Longleftrightarrow\ f(B(z_0,\varepsilon))\supset B(f(z_0),\delta)$.

Remark. If f is injective on Ω, then $\forall \varepsilon>0, \exists\delta>0$ $f^{-1}(B(f(z_0),\delta))\subset B(z_0,\varepsilon)$,
so f^{-1} is continuous. $\left(\forall\varepsilon>0\ \exists\delta>0: f^{-1}(B(w_0,\delta))\subset f(B(f^{-1}(w_0),\varepsilon))\right)$
$z_0=f^{-1}(w_0)$

Proof. Take $\tilde{\varepsilon}=\min(\varepsilon,\varepsilon_0)$ from Local Map Theorem.
Then, by the theorem $f(B(z_0,\varepsilon))\supset f(B(z_0,\tilde\varepsilon))\supset B(f(z_0),\delta)$,
for δ from Theorem ∎

Corollary. (Border correspondence).

S - closed, bounded, $\boxed{f\in A(S),\ f\not\equiv \text{const}}$. Then $f(\partial S)\supset \partial(f(S))$.

Proof. $w_0 \in f(\mathrm{Int}(S)) \Rightarrow w_0\in \mathrm{Int}(f(S))$ (open → open).

So $w_0\in\partial f(S) \Rightarrow w_0 \notin f(\mathrm{Int}(S))$. But $f(S)$ - compact, so closed.

So $w_0 \in \partial f(S) \Rightarrow$ $w_0\in f(S)$, $w_0\notin f(\mathrm{Int}\, S)$ $\Rightarrow w_0\in f(\partial S)$ ∎

Theorem. Let f be a 1-1 analytic function $f:\Omega\to\mathbb{C}$.
Then $f^{-1}: f(\Omega)\to\Omega$ is also analytic.

Proof. If Ω is a region, so is $f(\Omega)$ - it is open (f is open) and connected.
$f'(z)\neq 0\ \forall z\in\Omega$ (by local behavior). By open map theorem, f^{-1} is continuous.
So, by a homework problem, f^{-1} is complex differentiable ∎

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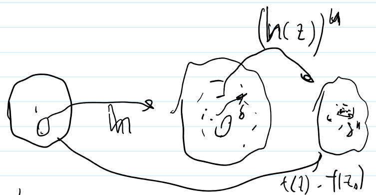
Local coordinate change. Let $f(z)\in A(\Omega)$, $z_0\in\Omega$, $f(z)-f(z_0)$ has zero of order n at z_0. Then $\exists\,\varepsilon>0$ and a conformal (1-1) $h\in A(B(z_0,\varepsilon))$: $f(z)-f(z_0)=(h(z))^n$, $h(z_0)=0$.

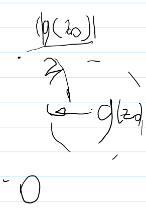


Proof. $f(z)-f(z_0)=(z-z_0)^n g(z)$ for some $g(z)\in A(\Omega)$, $g(z_0)\neq 0$. Fix $0<r<\mathrm{dist}(z_0,\partial\Omega)$ such that $|z-z_0|\le r \Rightarrow f(z)\neq f(z_0)$.

Let $\gamma=\{|z-z_0|=r\}$, oriented counterclockwise.
Then $n(f(\gamma), f(z_0))=n$

$\exists\,\varepsilon>0$ $(r>\varepsilon)$: $|z-z_0|<\varepsilon \Rightarrow$ 1) $z\in\Omega$ 2) $|g(z)-g(z_0)|<\frac{|g(z_0)|}{2}$ 3) $|f(z)-f(z_0)|<\mathrm{dist}(f(z_0), f(\gamma))$ ($\Rightarrow n(f(\gamma), f(z))=n$)

A branch $\ell(w)$ of $\log w$ is defined in $B\left(g(z_0), \frac{|g(z_0)|}{2}\right)$.



$$f(z)-f(z_0)=(z-z_0)^n\exp(\ell(g(z)))$$

So the function $h(z):=(z-z_0)\exp\left(\frac{\ell(g(z))}{n}\right)$ is well-defined in $B(z_0,\varepsilon)$, analytic in $B(z_0,\varepsilon)$ and satisfies "$h(z)^n = (z-z_0)^n\cdot\left(\exp\left(\frac{\ell(g(z))}{n}\right)\right)^n = (z-z_0)^n\exp(\ell(g(z))) = (z-z_0)^n g(z) = f(z)-f(z_0)$.

Note now that for any $z\in B(z_0,\varepsilon)$, since $f(z)-f(z_0)=h(z)^n$

$n = n(f(\gamma), f(z)) = n\cdot n(h(\gamma), h(z))$, so $n(h(\gamma),h(z))=1$, which means that if $z'\neq z$, $|z'-z_0|<\varepsilon \Rightarrow h(z)\neq h(z')$.
So h is conformal ∎

argument principle $\frac{1}{2\pi i}\oint_\gamma \frac{f'(\zeta)}{f(\zeta)-f(z)} = n\oint_\gamma\frac{h'(\zeta)}{h(\zeta)-h(z)}d\zeta$

$((f(\zeta)-f(z_0)=(h(\zeta)-h(z_0))^n))$

$n(f(\gamma), f(z_0)) = n(h(\gamma), h(z_0))\cdot n$

Theorem (Maximum Principle). Let $f \in A(\Omega)$, $z_0 \in \Omega$ and $|f|$ reaches a local maximum at z_0 (i.e. $\exists \varepsilon > 0 : |z - z_0| < \varepsilon, z \in \Omega \Rightarrow |f(z)| \le |f(z_0)|$).

Then $f \equiv \text{const}$.

Proof. Assume $f \not\equiv$ const. Take $B(z_0,\varepsilon) \subset \Omega$. Then $\exists \delta > 0$ $f(\underbrace{B(z_0,\varepsilon)}_{\text{open!}}) \supset B(f(z_0),\delta)$. $f(z_0) \ne 0$.

So $f(z_0) + \frac{\delta}{2}\frac{f(z_0)}{|f(z_0)|} \in B(f(z_0),\delta) \subset f(B(z_0,\varepsilon))$, so

$\exists z : |z - z_0| < \varepsilon,\ z \in \Omega,\ f(z) = f(z_0) + \frac{\delta}{2}\frac{f(z_0)}{|f(z_0)|}$.

$|f(z)| = \left(1 + \frac{\delta}{2|f(z_0)|}\right)|f(z_0)| > |f(z_0)|$ — contradiction!

How to modify it for $f(z_0) = 0$? $z : f(z) = \frac{\delta}{2} \in B(0,\delta)$

$|f(z)| > 0 = |f(z_0)|$.

Other way: $|f(z)| \le 0$ in $B(z_0,\varepsilon) \Rightarrow f \equiv 0$ in $B(z_0,\varepsilon) \underset{\text{Uniqueness thm.}}{\Rightarrow} f \equiv 0$ in Ω.

Theorem. Let S be closed and bounded.
$f \in C(S)$ – continuous on S.
$f \in A(\operatorname{Int} S)$. Then

$$\max_{z \in S} |f(z)| = \max_{z \in \partial S} |f(z)|.$$

If $f \not\equiv$ const, then $\forall z \in \operatorname{Int} S$, $|f(z)| < \max\limits_{z \in S} |f(z)| = \max\limits_{z \in \partial S} |f(z)|$

Proof. If $f \equiv$ const – nothing to prove.
If $f \not\equiv$ const, by compactness, $\exists z_0 \in S : f(z_0) = \max\limits_{z \in S} |f(z)|$
By Maximum Principle, $z_0 \notin \operatorname{Int} S$.
So $z_0 \in \partial S$, and $\forall z \in \operatorname{Int} S$, $|f(z)| < |f(z_0)|$ – again, by Maximum Principle.

Yet another proof of FTA.

Let $p(z) = a_n z^n + \dots + a_0$, $a_n \ne 0$.
Assume: $\forall z : p(z) \ne 0$.
Consider $f(z) := \frac{1}{p(z)}$ – analytic.
Then $\forall |z| < R$, $|f(z)| \le \max\limits_{|z|=R} |f(z)| = \frac{1}{\min\limits_{|z|=R} |p(z)|} =: m_R$ $\quad \overline{B(0,R)}$
But as $|z| \to \infty$, $|p(z)| = |z^n|\left|\left(a_n + \frac{a_{n-1}}{z} + \dots + \frac{a_0}{z^n}\right)\right| \to \lim\limits_{|z|\to\infty} |z^n| \cdot |a_n| = \infty$, $a_n \ne 0$
So as $R \to \infty$, $m_R \to 0$. So $\forall z$ $|f(z)| = 0$ — contradiction ∎